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Nonlinear dynamics of elastic rods using the Cosserat theory: Modelling and simulation

D.Q. Cao ^{a,*,1}, Robin W. Tucker ^b^a School of Astronautics, Harbin Institute of Technology, P.O. Box 137, Harbin 150001, China^b Department of Physics, Lancaster University, Lancaster LA1 4YB, UK

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Abstract

The method of Cosserat dynamics is employed to explore the nonplanar nonlinear dynamics of elastic rods. The rod, which is assumed to undergo flexure about two principal axes, extension, shear and torsion, are described by a general geometrically exact theory. Based on the Cosserat theory, a set of governing partial differential equations of motion with arbitrary boundary conditions is formulated in terms of the displacements and angular variables, thus the dynamical analysis of elastic rods can be carried out rather simply. The case of doubly symmetric cross-section of the rod is considered and the Kirchhoff constitutive relations are adopted to provide an adequate description of elastic properties in terms of a few elastic moduli. A cantilever is given as a simple example to demonstrate the use of the formulation developed. The nonlinear dynamic model with the corresponding boundary and initial conditions are numerically solved using the Femlab/Matlab software packages. The corresponding nonlinear dynamical responses of the cantilever under external harmonic excitations are presented through numerical simulations.

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1. Introduction

Nonlinear dynamic analysis of slender elastic structures under external forces and torques and parametric excitations remains an active area of study. Such a study can find application in accelerating missiles and space crafts, components of high-speed machinery, manipulator arm, microelectronic mechanical structures (MEMS), components of bridges (such as towers and cables) and other structural elements. Considerable attention has been devoted to the study of nonlinear dynamics of rods or beams subject to both external and parametric excitations (see, for example, Nayfeh and D.T.Mook, 1979; Saito and Koizumi, 1982; Kar and Dwivedy, 1999 and the references cited therein). While attention so far has mainly been devoted to the

* Corresponding author. Tel.: +86 (0) 451 86414479; fax: +86 (0) 451 86402822.

E-mail addresses: dqcao@hit.edu.cn (D.Q. Cao), r.tucker@lancaster.ac.uk (R.W. Tucker).

¹ On leave from the Department of Physics, Lancaster University, Lancaster LA1 4YB, UK.

study of planar, nonlinear dynamic analysis of beams, research has been done concerning nonplanar, nonlinear motions of beams. Bolotin (1964) addressed such motions in beams, but restricted himself to consideration of nonlinear inertia terms and stability of the planar response. Crespo da Silva and Glynn (1978a) formulated the equations of motion describing the nonplanar, nonlinear dynamics of an inextensional beam. The nonplanar, nonlinear forced oscillations of a cantilever are then analyzed in Crespo da Silva and Glynn (1978b) using the perturbation method. Cartmell (1990) and Forehand and Cartmell (2001) derived the nonlinear equation of motion for the in-plane and out-of-plane forced vibration of cantilever beams with a lumped mass.

The studies mentioned above are restricted to systems with no extension of the beam's neutral axis and no warping or shear deformation. Crespo da Silva (1988a,b) investigated the problem of nonlinear dynamics of the nonplanar flexural-torsional-extensional beams, but the effects of rotary inertia and shear deformation were neglected. Thus, cross-sectional dimensions of the beam were assumed to be small enough in comparison to the beam length. However, the shear deformation may be of considerable importance and can not be negligible for studying the vibration of high frequencies when a comparative short rod is investigated. In such a case, the effect of shear deformation should be taken into account for. For the planar problem, although such effects can be included by using the Timoshenko beam theory, most of the studies are limited to the determination of natural frequencies and eigenfunctions (Huang, 1961; Bishop and Price, 1977; Grant, 1978; Bruch and Mitchell, 1987). For the three-dimensional problem, we refer to Simo and Vu-Quoc (1988, 1991) for a geometrically exact rod model incorporating shear and torsion-warping deformation. Further studies on the dynamic formulation of sandwich beams have been presented in Vu-Quoc and Deng (1995) and Vu-Quoc and Ebcioğlu (1995) based on the geometrically-exact description of the kinematics of deformation. Moreover, Esmailzadeh and Jalili (1998) investigated the parametric response of cantilever Timoshenko beams with lumped mass, but restricted themselves to consideration of nonlinear inertia terms. Based on the geometrically-exact model of sliding beams, parametric resonance has been presented in Vu-Quoc and Li (1995), where the beams can undergo large deformation with shear deformation accounted for.

In the case involving the full dynamic response, the strong nonlinearity and fully coupling introduce a challenge for solving the partial differential equations of motion of rods. We refer to Rubin (2001) for a formulation of a numerical solution procedure for three-dimensional dynamic analysis of rods by modelling the rod as a set of connected Cosserat points, also Rubin and Tufekci (2005) and Rubin (2000) for three-dimensional dynamics of a circular arch and shells using the theory of a Cosserat point, respectively. A Galerkin projection has been applied to discretize the governing partial differential equations of sliding beams in Vu-Quoc and Li (1995) and sandwich beams in Vu-Quoc and Deng (1997). Recently, a new modelling strategy has been proposed in Cao et al. (2006) to discretize the rod and to derive the ordinary differential equations of motion with third order nonlinear generic nodal displacements. This modelling strategy has been successfully used to investigate the nonlinear dynamics of typical MEMS device that comprises a resonator mass supported by four flexible beams (Cao et al., 2005).

In this paper we explore the nonplanar, nonlinear dynamics of an extensional shearable rod by using the simple Cosserat model. The method of Cosserat dynamics for elastic structures is employed since it can accommodate to a good approximation the nonlinear behavior of complex elastic structures composed of materials with different constitutive properties, variable geometry and damping characteristics (Antman, 1991; Antman et al., 1998; Tucker and Wang, 1999; Cull et al., 2000; Gratus and Tucker, 2003). With arbitrary boundary conditions, the Cosserat theory is used to formulate a set of governing partial differential equations of motion in terms of the displacements and angular variables, describing the nonplanar, nonlinear dynamics of an extensional rod. Bending about two principal axes, extension, shear and torsion are considered, and care is taken into account for all the nonlinear terms in the resulting equations. As an example, a simple cantilever is given to demonstrate the use of the formulation developed. The nonlinear dynamic model with the corresponding boundary and initial conditions are numerically solved using the commercially available software packages Femlab and Matlab. Corresponding nonlinear dynamical responses of the cantilever under external harmonic excitations are presented and discussed through numerical simulations.

The following conventions and nomenclature will be used through out this paper. Vectors, which are elements of Euclidean 3-space \mathcal{R}^3 , are denoted by lowercase, bold-face symbols, e.g., \mathbf{u} , \mathbf{v} ; vector-valued functions are denoted by lowercase, italic, bold-face symbols, e.g., \mathbf{u} , \mathbf{v} ; tensors are denoted by upper-case, bold-face symbols, e.g., \mathbf{I} , \mathbf{J} ; matrices are denoted by upper-case, italic, bold-face symbols, e.g., \mathbf{M} , \mathbf{K} . The symbols

∂_t and ∂_s denote differentiation with respect to time t and arc-length parameter s , respectively. The symbols $(\dot{\cdot})$ and (\prime) denote differentiation with respect to dimensionless time parameter τ and dimensionless length parameter σ , respectively.

2. Background on the simple Cosserat model

In this section the basic concepts of the Cosserat theory for an elastic rod of unstressed length ℓ are summarized. The three vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are assumed to form a fixed right-handed orthogonal basis. Elements of the rod are labelled in terms of the Lagrangian coordinate $0 \leq s \leq \ell$ at time t . The motion of an elastic segment may be described in terms of the motion in space of a vector \mathbf{r} that locates the line of centroids (shown dash-dotted) of the cross-sections as shown in Fig. 1. Specifying a unit vector \mathbf{d}_3 (which may be identified with the normal to the cross-section) at each point along this line enables the state of shear to be related to the angle between this vector and the tangent $\partial_s \mathbf{r}$ to the centroid space-curve. Specifying a second vector \mathbf{d}_2 orthogonal to the first vector (thereby placing it in the plane of the cross-section) can be used to encode the state of bending and twist along the element. In such a way, the current orientation of each cross-section at $s \in [0, \ell]$ is defined by specifying the orientation of a moving basis $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3 = \mathbf{d}_1 \times \mathbf{d}_2$, see Simo and Vu-Quoc (1988) and Vu-Quoc and Deng (1995) for details. Elastic deformations about the line of centroids are then coded into the rates of change of \mathbf{r} and the triad $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$ of the cross-section at s . Thus a time dependent field of two mutually orthogonal unit vectors along the segment provides three continuous dynamical degrees of freedom that, together with the three continuous degrees of freedom describing the centroid space-curve relative to some arbitrary origin in space (with fixed inertial frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$), define a simple Cosserat rod model. Supplemented with appropriate constitutive relations and boundary conditions such a model can fully accommodate the modes of vibration that are traditionally associated with the motion of slender rods in the engineering literature: namely axial motion along its length, torsional or rotational motion and transverse or lateral motion.

The tangent to the space-curve, $\mathbf{v} = \mathbf{v}(s, t)$, is given by

$$\mathbf{v}(s, t) = \partial_s \mathbf{r}(s, t), \quad (1)$$

the rotation of the directions along the space curve and the local angular velocity vector of the director triad are measured by $\mathbf{u} = \mathbf{u}(s, t)$, $\mathbf{w} = \mathbf{w}(s, t)$ according to

$$\begin{cases} \partial_s \mathbf{d}_i(s, t) = \mathbf{u}(s, t) \times \mathbf{d}_i(s, t), \\ \partial_t \mathbf{d}_i(s, t) = \mathbf{w}(s, t) \times \mathbf{d}_i(s, t), \end{cases} \quad (2)$$

respectively. It follows from the first equation of (2) that

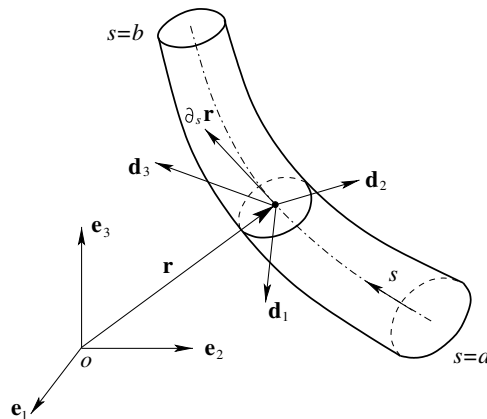


Fig. 1. A simple Cosserat model.

$$\sum_{i=1}^3 (\mathbf{d}_i \times \partial_s \mathbf{d}_i) = \sum_{i=1}^3 (\mathbf{d}_i \times (\mathbf{u} \times \mathbf{d}_i)) = \sum_{i=1}^3 (\mathbf{u}(\mathbf{d}_i \cdot \mathbf{d}_i) - \mathbf{d}_i(\mathbf{d}_i \cdot \mathbf{u})) = 2\mathbf{u}. \quad (3)$$

Similarly, we have

$$\sum_{i=1}^3 (\mathbf{d}_i \times \partial_t \mathbf{d}_i) = \sum_{i=1}^3 (\mathbf{d}_i \times (\mathbf{w} \times \mathbf{d}_i)) = \sum_{i=1}^3 (\mathbf{w}(\mathbf{d}_i \cdot \mathbf{d}_i) - \mathbf{d}_i(\mathbf{d}_i \cdot \mathbf{w})) = 2\mathbf{w}. \quad (4)$$

Since the basis $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ is natural for the intrinsic description of deformation, we decompose relevant vector valued functions with respect to it:

$$\begin{cases} \mathbf{v}(s, t) = \sum_{i=1}^3 v_i(s, t) \mathbf{d}_i(s, t), \\ \mathbf{u}(s, t) = \sum_{i=1}^3 u_i(s, t) \mathbf{d}_i(s, t), \\ \mathbf{w}(s, t) = \sum_{i=1}^3 w_i(s, t) \mathbf{d}_i(s, t). \end{cases} \quad (5)$$

The contact forces $\mathbf{n}(s, t)$ and contact torques $\mathbf{m}(s, t)$ are related to the extension and shear strains $\mathbf{v}(s, t)$, and flexure and torsion strains $\mathbf{u}(s, t)$ by constitutive relations, respectively. The angular momentum $\mathbf{h}(s, t)$ is related to the rotary inertia and the director angular velocity $\mathbf{w}(s, t)$.

The dynamical evolution of the rod with density, $\rho(s)$, and cross-section area, $A(s)$ is governed by the Newton's dynamical laws:

$$\begin{cases} \rho(s)A(s)\partial_t \mathbf{r} = \partial_s \mathbf{n}(s, t) + \mathbf{f}(s, t), \\ \partial_t \mathbf{h}(s, t) = \partial_s \mathbf{m}(s, t) + \mathbf{v}(s, t) \times \mathbf{n}(s, t) + \mathbf{l}(s, t), \end{cases} \quad (6)$$

where $\mathbf{f}(s, t)$ and $\mathbf{l}(s, t)$ denote external force and torque densities, respectively, the contact force vector, contact torque vector, and the angular momentum vector can be written as

$$\begin{cases} \mathbf{n}(s, t) = \sum_{i=1}^3 n_i(s, t) \mathbf{d}_i(s, t), \\ \mathbf{m}(s, t) = \sum_{i=1}^3 m_i(s, t) \mathbf{d}_i(s, t), \end{cases} \quad (7)$$

and

$$\mathbf{h}(s, t) = \sum_{i=1}^3 h_i(s, t) \mathbf{d}_i(s, t), \quad (8)$$

respectively. For more details about Eqs. (6)–(8), we refer to the governing equations (i)–(iii) derived in terms of the noncommutative Lie group of proper orthogonal transformations in [Simo and Vu-Quoc \(1988\)](#), see also in [Vu-Quoc and Deng \(1995\)](#) and [Vu-Quoc and Ebcioglu \(1995\)](#). It will be assumed that the Young's modulus E , the shear modulus G , and the material density ρ along the rod are functions of s only, and the mass center coincides with the area centroid of the cross-section at s . In this case, the simplest constitutive model for the linear elastic material is based on the Kirchhoff constitutive relations with shear deformation, see the discussions in [Simo and Vu-Quoc \(1988, 1991\)](#) for details of constitutive relations. Such a model provide an adequate description of elastic properties in terms of a few elastic moduli. The presence of arbitrary rotations relating the local director frame to the global inertial frame renders the equations of motion inherently nonlinear. The contact forces, contact torques and the angular momentum are then given as

$$\mathbf{n} = \mathbf{K}(\mathbf{v} - \mathbf{d}_3), \quad \mathbf{m} = \mathbf{J}\mathbf{u}, \quad \mathbf{h} = \mathbf{I}\mathbf{w}, \quad (9)$$

where, for a symmetric cross-section of a rod, the tensors \mathbf{K} , \mathbf{J} and \mathbf{I} are described as

$$\begin{cases} \mathbf{K}(s, t) = \sum_{i=1}^3 K_{ii}(s, t)(\mathbf{d}_i(s, t) \otimes \mathbf{d}_i(s, t)), \\ \mathbf{J}(s, t) = \sum_{i=1}^3 J_{ii}(s, t)(\mathbf{d}_i(s, t) \otimes \mathbf{d}_i(s, t)), \\ \mathbf{I}(s, t) = \sum_{i=1}^3 I_{ii}(s, t)(\mathbf{d}_i(s, t) \otimes \mathbf{d}_i(s, t)). \end{cases} \quad (10)$$

Here the symbol \otimes is used to denote the tensorial product $\mathbf{u} \otimes \mathbf{v}$ of two vectors \mathbf{u} and \mathbf{v} that assigns to each vector \mathbf{w} the vector $(\mathbf{u} \otimes \mathbf{v})\mathbf{w} = (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$. The corresponding terms in (10) can be written as

$$\begin{cases} K_{11} = K_{22} = \kappa(s)G(s)A(s), & K_{33} = E(s)A(s), \\ J_{11} = \int_{A(s)} E(s)\eta^2 dA, & J_{22} = \int_{A(s)} E(s)\xi^2 dA, \\ I_{11} = \int_{A(s)} \rho(s)\eta^2 dA, & I_{22} = \int_{A(s)} \rho(s)\xi^2 dA, \\ J_{33} = \int_{A(s)} G(s)\left(\xi^2 + \eta^2 + \xi \frac{\partial \Psi}{\partial \eta} + \eta \frac{\partial \Psi}{\partial \xi}\right) dA, \\ I_{33} = \int_{A(s)} \rho(s)(\xi^2 + \eta^2) dA, \end{cases} \quad (11)$$

where Ψ is the warping function which is the same for all cross-sections according to the Saint-Venant Theory, and $\kappa(s)$ is a numerical factor depending on the shape of the cross-section at s .

Making use of (5), (9) and (10), we have

$$\begin{cases} \mathbf{n} = K_{11}v_1\mathbf{d}_1 + K_{22}v_2\mathbf{d}_2 + K_{33}(v_3 - 1)\mathbf{d}_3, \\ \mathbf{m} = J_{11}u_1\mathbf{d}_1 + J_{22}u_2\mathbf{d}_2 + J_{33}u_3\mathbf{d}_3, \\ \mathbf{h} = I_{11}w_1\mathbf{d}_1 + I_{22}w_2\mathbf{d}_2 + I_{33}w_3\mathbf{d}_3. \end{cases} \quad (12)$$

Comparing with (7), we get

$$\begin{cases} n_1 = K_{11}v_1, & m_1 = J_{11}u_1, & h_1 = I_{11}w_1, \\ n_2 = K_{22}v_2, & m_2 = J_{22}u_2, & h_2 = I_{22}w_2, \\ n_3 = K_{33}(v_3 - 1), & m_3 = J_{33}u_3, & h_3 = I_{33}w_3. \end{cases} \quad (13)$$

3. Specifications for the deformed configuration space

Consider a uniform and initially straight rod of constant length ℓ , supported in an arbitrary manner at each end as shown in Fig. 2. It is assumed that the static equilibrium of the rod corresponds to the situation where the directions of \mathbf{d}_3 and \mathbf{e}_3 coincide, \mathbf{d}_1 and \mathbf{d}_2 are parallel to \mathbf{e}_1 and \mathbf{e}_2 , respectively.

The components of the elastic displacement vector of the centroid at an arbitrary location s is assumed to be $x(s, t)$, $y(s, t)$, $z(s, t)$, i.e.

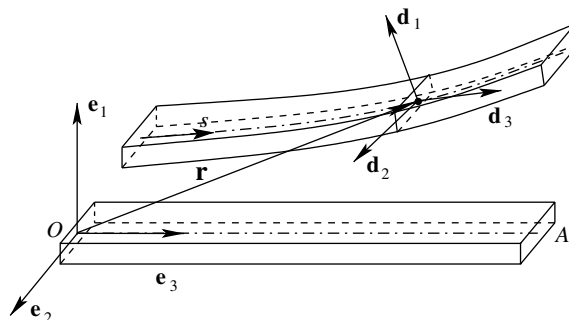


Fig. 2. Coordinate systems used in the development of governing equations.

$$\mathbf{r}(s, t) = x(s, t)\mathbf{e}_1 + y(s, t)\mathbf{e}_2 + (s + z(s, t))\mathbf{e}_3. \quad (14)$$

The orientation of the directors $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$ at location s , relative to the inertial basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, can be described by three successive rotations, called Euler angles. Alternatively, the moving frame can be described by the rotation group $\text{SO}(3)$ as the formulations in Simo and Vu-Quoc (1988, 1991), Vu-Quoc and Deng (1995) and Vu-Quoc and Ebcioğlu (1995). The three successive rotations starts by rotating the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ an angle $\psi(s, t)$ about the axis aligned with the director \mathbf{e}_2 as shown in Fig. 3a. Next, we rotate the basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ an angle $\theta(s, t)$ about the axis aligned with the new director $\hat{\mathbf{e}}_1$ as shown in Fig. 3b. Finally, we rotate the basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{d}}_3\}$ an angle $\phi(s, t)$ about the axis aligned with the new director $\hat{\mathbf{d}}_3$ as shown in Fig. 3c.

The representative matrices for the three successive rotations can be written as

$$\mathbf{D}_\psi(s, t) = \begin{bmatrix} \cos \psi(s, t) & 0 & -\sin \psi(s, t) \\ 0 & 1 & 0 \\ \sin \psi(s, t) & 0 & \cos \psi(s, t) \end{bmatrix}, \quad (15)$$

$$\mathbf{D}_\theta(s, t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta(s, t) & \sin \theta(s, t) \\ 0 & -\sin \theta(s, t) & \cos \theta(s, t) \end{bmatrix}, \quad (16)$$

and

$$\mathbf{D}_\phi(s, t) = \begin{bmatrix} \cos \phi(s, t) & \sin \phi(s, t) & 0 \\ -\sin \phi(s, t) & \cos \phi(s, t) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (17)$$

respectively.

Denote $d_{ij} = \mathbf{d}_i \cdot \mathbf{e}_j$. Then, using Eqs. (15)–(17), one can write the transformation matrix between the inertial basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and the basis $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ as follows:

$$\begin{aligned} \mathbf{D}(s, t) &= [d_{ij}(s, t)] = \mathbf{D}_\phi(s, t)\mathbf{D}_\theta(s, t)\mathbf{D}_\psi(s, t) \\ &= \begin{bmatrix} \cos \phi \cos \psi + \sin \phi \sin \theta \sin \psi & \sin \phi \cos \theta & -\cos \phi \sin \psi + \sin \phi \sin \theta \cos \psi \\ -\sin \phi \cos \psi + \cos \phi \sin \theta \sin \psi & \cos \phi \cos \theta & \sin \phi \sin \psi + \cos \phi \sin \theta \cos \psi \\ \cos \theta \sin \psi & -\sin \theta & \cos \theta \cos \psi \end{bmatrix}. \end{aligned} \quad (18)$$

Since \mathbf{D} is an orthogonal matrix, from the definition of $d_{ij} = \mathbf{d}_i \cdot \mathbf{e}_j$, we have

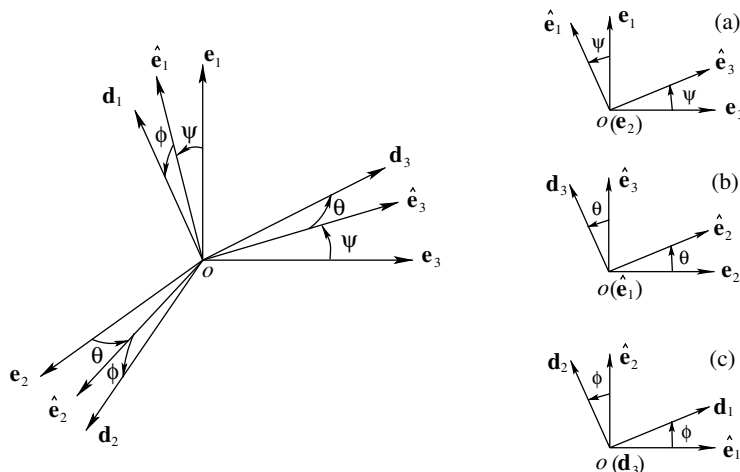


Fig. 3. The successive angle rotations of a differential element of arc length ds .

$$\mathbf{d}_i = \sum_{j=1}^3 d_{ij} \mathbf{e}_j, \quad \mathbf{e}_j = \sum_{k=1}^3 d_{kj} \mathbf{d}_k. \quad (19)$$

Thus

$$\partial_s \mathbf{d}_i = \partial_s \left(\sum_{j=1}^3 d_{ij} \mathbf{e}_j \right) = \sum_{j=1}^3 \partial_s d_{ij} \mathbf{e}_j = \sum_{j=1}^3 \sum_{k=1}^3 (\partial_s d_{ij}) d_{kj} \mathbf{d}_k. \quad (20)$$

Substituting (20) into (3) yields

$$\begin{aligned} \mathbf{u} &= \frac{1}{2} \sum_{i=1}^3 \mathbf{d}_i \times \partial_s \mathbf{d}_i = \frac{1}{2} \sum_{i=1}^3 \mathbf{d}_i \times \left(\sum_{j=1}^3 \sum_{k=1}^3 (\partial_s d_{ij}) d_{kj} \mathbf{d}_k \right) \\ &= \frac{1}{2} \sum_{j=1}^3 [(\partial_s d_{1j}) d_{2j} - (\partial_s d_{2j}) d_{1j}] \mathbf{d}_3 + \frac{1}{2} \sum_{j=1}^3 [(\partial_s d_{2j}) d_{3j} - (\partial_s d_{3j}) d_{2j}] \mathbf{d}_1 \\ &\quad + \frac{1}{2} \sum_{j=1}^3 [(\partial_s d_{3j}) d_{1j} - (\partial_s d_{1j}) d_{3j}] \mathbf{d}_2. \end{aligned} \quad (21)$$

Taking notice of

$$\sum_{j=1}^3 d_{ij}(s, t) d_{kj}(s, t) = \delta_{ik}, \quad (22)$$

where δ is the Kronecker δ -function, we have

$$\sum_{j=1}^3 [(\partial_s d_{ij}(s, t)) d_{kj}(s, t) + d_{ij}(s, t) (\partial_s d_{kj}(s, t))] = 0. \quad (23)$$

Hence, (21) can be written as

$$\mathbf{u} = \sum_{j=1}^3 [(\partial_s d_{2j}) d_{3j} \mathbf{d}_1 + (\partial_s d_{3j}) d_{1j} \mathbf{d}_2 + (\partial_s d_{1j}) d_{2j} \mathbf{d}_3]. \quad (24)$$

Therefore, from (18) and (24) we can obtain

$$\begin{cases} u_1 = \partial_s \theta \cos \phi + \partial_s \psi \cos \theta \sin \phi, \\ u_2 = -\partial_s \theta \sin \phi + \partial_s \psi \cos \theta \cos \phi, \\ u_3 = \partial_s \phi - \partial_s \psi \sin \theta. \end{cases} \quad (25)$$

Similarly, replacing the time derivative by the space derivative, we can easily obtain

$$\begin{cases} w_1 = \partial_t \theta \cos \phi + \partial_t \psi \cos \theta \sin \phi, \\ w_2 = -\partial_t \theta \sin \phi + \partial_t \psi \cos \theta \cos \phi, \\ w_3 = \partial_t \phi - \partial_t \psi \sin \theta. \end{cases} \quad (26)$$

In order to express the extension and shear strains as functions of displacements $x(s, t)$, $y(s, t)$, $z(s, t)$ and angles $\theta(s, t)$, $\psi(s, t)$, $\phi(s, t)$, we define

$$r_1^e(s, t) = x(s, t), \quad r_2^e(s, t) = y(s, t), \quad r_3^e(s, t) = s + z(s, t). \quad (27)$$

Then, the strain vector $\mathbf{v}(s, t)$ can be written as

$$\mathbf{v} = \partial_s \mathbf{r} = \sum_{i=1}^3 (\partial_s r_i^e) \mathbf{e}_i = \sum_{i=1}^3 (\partial_s r_i^e) d_{ji} \mathbf{d}_j. \quad (28)$$

Substituting (18) and (27) into (28) yields

$$\begin{cases} v_1 = \partial_s x (\cos \phi \cos \psi + \sin \phi \sin \theta \sin \psi) + \partial_{sy} \sin \phi \cos \theta \\ \quad + (1 + \partial_{sz}) (-\cos \phi \sin \psi + \sin \phi \sin \theta \cos \psi), \\ v_2 = \partial_s x (-\sin \phi \cos \psi + \cos \phi \sin \theta \sin \psi) + \partial_{sy} \cos \phi \cos \theta \\ \quad + (1 + \partial_{sz}) (\sin \phi \sin \psi + \cos \phi \sin \theta \cos \psi), \\ v_3 = \partial_s x \cos \theta \sin \psi - \partial_{sy} \sin \theta + (1 + \partial_{sz}) \cos \theta \cos \psi. \end{cases} \quad (29)$$

4. Governing equations of motion

To obtain the partial differential equations of motion in terms of the displacements $x(s, t)$, $y(s, t)$, $z(s, t)$ and angles $\theta(s, t)$, $\psi(s, t)$, $\phi(s, t)$, we decompose the contact force $\mathbf{n}(s, t)$ with respect to the fixed basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as

$$\mathbf{n}(s, t) = \sum_{i=1}^3 n_i^e(s, t) \mathbf{e}_i. \quad (30)$$

Then, making use of (9) and (19), we have

$$\mathbf{n} = \sum_{i=1}^3 n_i \mathbf{d}_i = \sum_{i=1}^3 \sum_{j=1}^3 n_i d_{ij} \mathbf{e}_j = \sum_{j=1}^3 n_j^e \mathbf{e}_j. \quad (31)$$

It follows from (13) that

$$\begin{cases} n_1^e = \sum_{i=1}^3 d_{i1} n_i = d_{11} K_{11} v_1 + d_{21} K_{22} v_2 + d_{31} K_{33} (v_3 - 1), \\ n_2^e = \sum_{i=1}^3 d_{i2} n_i = d_{12} K_{11} v_1 + d_{22} K_{22} v_2 + d_{32} K_{33} (v_3 - 1), \\ n_3^e = \sum_{i=1}^3 d_{i3} n_i = d_{13} K_{11} v_1 + d_{23} K_{22} v_2 + d_{33} K_{33} (v_3 - 1). \end{cases} \quad (32)$$

To proceed, using the relations between the strain vector $\mathbf{u}(s, t)$ and the differentiation of the directors $\mathbf{d}_1(s, t)$, $\mathbf{d}_2(s, t)$ and $\mathbf{d}_3(s, t)$ with respect to s , and the relations between the angular velocity vector $\mathbf{w}(s, t)$ and the differentiation of the directors $\mathbf{d}_1(s, t)$, $\mathbf{d}_2(s, t)$ and $\mathbf{d}_3(s, t)$ with respect to t , we can easily obtain the following expressions:

$$\begin{aligned} \partial_s \mathbf{m} &= \sum_{i=1}^3 \partial_s (m_i \mathbf{d}_i) = \sum_{i=1}^3 (\partial_s m_i \mathbf{d}_i + m_i \mathbf{u} \times \mathbf{d}_i) \\ &= (\partial_s m_1 - m_2 u_3 + m_3 u_2) \mathbf{d}_1 \\ &\quad + (\partial_s m_2 - m_3 u_1 + m_1 u_3) \mathbf{d}_2 \\ &\quad + (\partial_s m_3 - m_1 u_2 + m_2 u_1) \mathbf{d}_3, \end{aligned} \quad (33)$$

$$\begin{aligned} \partial_t \mathbf{h} &= \sum_{i=1}^3 \partial_t (h_i \mathbf{d}_i) = \sum_{i=1}^3 (\partial_t h_i \mathbf{d}_i + h_i \mathbf{w} \times \mathbf{d}_i) \\ &= (\partial_t h_1 - h_2 w_3 + h_3 w_2) \mathbf{d}_1 \\ &\quad + (\partial_t h_2 - h_3 w_1 + h_1 w_3) \mathbf{d}_2 \\ &\quad + (\partial_t h_3 - h_1 w_2 + h_2 w_1) \mathbf{d}_3. \end{aligned} \quad (34)$$

In addition, we record

$$\mathbf{v} \times \mathbf{n} = \left(\sum_{i=1}^3 v_i \mathbf{d}_i \right) \times \left(\sum_{j=1}^3 n_j \mathbf{d}_j \right) = (v_2 n_3 - v_3 n_2) \mathbf{d}_1 + (v_3 n_1 - v_1 n_3) \mathbf{d}_2 + (v_1 n_2 - v_2 n_1) \mathbf{d}_3. \quad (35)$$

Now, let us assume that the external force and torque are distributed loadings with fixed direction and prescribed intensity. Consequently, the external force and torque densities can be written as:

$$\begin{cases} \mathbf{f}(s, t) = f_x(s, t)\mathbf{e}_1 + f_y(s, t)\mathbf{e}_2 + f_z(s, t)\mathbf{e}_3, \\ \mathbf{l}(s, t) = l_x(s, t)\mathbf{e}_1 + l_y(s, t)\mathbf{e}_2 + l_z(s, t)\mathbf{e}_3. \end{cases} \quad (36)$$

Decomposing the torque density with respect to the moving basis $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$, we have

$$\mathbf{l}(s, t) = \sum_{i=1}^3 l_i(s, t)\mathbf{d}_i(s, t), \quad (37)$$

where, $l_i(s, t)$ ($i = 1, 2, 3$) can be expressed by using the transformation matrix (18) and the relation (19) as

$$\begin{cases} l_1 = l_x(\cos \phi \cos \psi + \sin \phi \sin \theta \sin \psi) + l_y \sin \phi \cos \theta \\ \quad + l_z(-\cos \phi \sin \psi + \sin \phi \sin \theta \cos \psi), \\ l_2 = l_x(-\sin \phi \cos \psi + \cos \phi \sin \theta \sin \psi) + l_y \cos \phi \cos \theta \\ \quad + l_z(\sin \phi \sin \psi + \cos \phi \sin \theta \cos \psi), \\ l_3 = l_x \cos \theta \sin \psi - l_y \sin \theta + l_z \cos \theta \cos \psi. \end{cases} \quad (38)$$

Substituting the expressions (32)–(37) into the dynamic equation (6), we can obtain

$$\begin{cases} \rho A \partial_{tt} x = \partial_s(d_{11}K_{11}v_1 + d_{21}K_{22}v_2 + d_{31}K_{33}(v_3 - 1)) + f_x, \\ \rho A \partial_{tt} y = \partial_s(d_{12}K_{11}v_1 + d_{22}K_{22}v_2 + d_{32}K_{33}(v_3 - 1)) + f_y, \\ \rho A \partial_{tt} z = \partial_s(d_{13}K_{11}v_1 + d_{23}K_{22}v_2 + d_{33}K_{33}(v_3 - 1)) + f_z, \\ \partial_t h_1 - h_2 w_3 + h_3 w_2 = \partial_s m_1 - m_2 u_3 + m_3 u_2 + v_2 n_3 - v_3 n_2 + l_1, \\ \partial_t h_2 - h_3 w_1 + h_1 w_3 = \partial_s m_2 - m_3 u_1 + m_1 u_3 + v_3 n_1 - v_1 n_3 + l_2, \\ \partial_t h_3 - h_1 w_2 + h_2 w_1 = \partial_s m_3 - m_1 u_2 + m_2 u_1 + v_1 n_2 - v_2 n_1 + l_3, \end{cases} \quad (39)$$

where d_{ij} is defined by (18), n_i , m_i and h_i are given by (13), while u_i , w_i , v_i and l_i are given by (25), (26), (29) and (38), respectively.

The boundary conditions at the ends of the rod should be determined in each particular case. For example, at a fixed end the deflections and the rotations are equal to zero. In this case the boundary conditions are

$$x = y = z = 0 \quad \text{and} \quad \theta = \psi = \phi = 0. \quad (40)$$

At a free end the boundary conditions are

$$\alpha_m \mathbf{n} + \mathbf{f}^b(t) = 0 \quad \text{and} \quad \alpha_m \mathbf{m} + \mathbf{l}^b(t) = 0, \quad (41)$$

where $\alpha_m = +1$ at the right end of the rod, $\alpha_m = -1$ at the left end, and

$$\begin{cases} \mathbf{f}^b(t) = f_x^b(t)\mathbf{e}_1 + f_y^b(t)\mathbf{e}_2 + f_z^b(t)\mathbf{e}_3, \\ \mathbf{l}^b(t) = l_x^b(t)\mathbf{e}_1 + l_y^b(t)\mathbf{e}_2 + l_z^b(t)\mathbf{e}_3 \end{cases} \quad (42)$$

are the external force and torque applied to the corresponding end of the rod. As the external distributed torque $\mathbf{l}(s, t)$, the torque vector $\mathbf{l}^b(t)$ can be expressed in the moving frame $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ and the corresponding components $l_i^b(t)$ ($i = 1, 2, 3$) can be determined through the relation (19) and the transformation matrix (18) as follows:

$$\begin{cases} l_1^b = l_x^b(\cos \phi \cos \psi + \sin \phi \sin \theta \sin \psi) + l_y^b \sin \phi \cos \theta \\ \quad + l_z^b(-\cos \phi \sin \psi + \sin \phi \sin \theta \cos \psi), \\ l_2^b = l_x^b(-\sin \phi \cos \psi + \cos \phi \sin \theta \sin \psi) + l_y^b \cos \phi \cos \theta \\ \quad + l_z^b(\sin \phi \sin \psi + \cos \phi \sin \theta \cos \psi), \\ l_3^b = l_x^b \cos \theta \sin \psi - l_y^b \sin \theta + l_z^b \cos \theta \cos \psi, \end{cases} \quad (43)$$

with $\psi = \psi(s^*, t)$, $\theta = \theta(s^*, t)$ and $\phi = \phi(s^*, t)$ at the boundary $s = s^*$ ($s^* = 0$ or ℓ).

Substituting the relations (32) and (42) into the first equation of (41), the force boundary conditions can be rewritten as

$$\begin{cases} \alpha_m[d_{11}K_{11}v_1 + d_{21}K_{22}v_2 + d_{31}K_{33}(v_3 - 1)] + f_x^b(t) = 0, \\ \alpha_m[d_{12}K_{11}v_1 + d_{22}K_{22}v_2 + d_{32}K_{33}(v_3 - 1)] + f_y^b(t) = 0, \\ \alpha_m[d_{13}K_{11}v_1 + d_{23}K_{22}v_2 + d_{33}K_{33}(v_3 - 1)] + f_z^b(t) = 0. \end{cases} \quad (44)$$

Similarly, the torque boundary conditions can be obtained as

$$\alpha_m J_{11}u_1 + l_1^b(t) = 0, \quad \alpha_m J_{22}u_2 + l_2^b(t) = 0, \quad \alpha_m J_{33}u_3 + l_3^b(t) = 0. \quad (45)$$

For simplicity, it will be assumed that the axes along the directors \mathbf{d}_1 , \mathbf{d}_2 and \mathbf{d}_3 are chosen to be the principal axes of inertia of the cross-section at s , and centered at the cross-section's center of mass. Then, we have

$$\begin{cases} K_{11} = \kappa GA, & K_{22} = \kappa GA, & K_{33} = EA; \\ J_{11} = E\gamma_{11}, & J_{22} = E\gamma_{22}, & J_{33} = G\gamma_{33}; \\ I_{11} = \rho\gamma_{11}, & I_{22} = \rho\gamma_{22}, & I_{33} = \rho(\gamma_{11} + \gamma_{22}), \end{cases} \quad (46)$$

where γ_{11} and γ_{22} are the moments of inertia of the rod cross-section, and γ_{33} is the torsional moment of inertia. For a rectangular cross-section, the torsional moment of inertia is as follows (Rubin, 2000):

$$\gamma_{33} = \frac{WH^3}{3} \left[1 - \frac{192H}{\pi^5 W} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^5} \tanh \frac{\pi(2n-1)W}{2H} \right] \quad \text{for } W \geq H,$$

where W and H are the dimensions of the cross-section. Moreover, the approximate formula of the torsional moment of inertia can be written as

$$\gamma_{33} = \frac{WH^3}{3} \left[1 - 0.63 \frac{H}{W} \left(1 - \frac{H^4}{12W^4} \right) \right] \quad \text{for } W \geq H.$$

As a prelude to expanding the nonlinear partial differential equations (39) to a form suitable for a perturbation analysis of the motion, it is useful to introduce some natural scales to obtain a dimensionless equation of motion. Introduce the dimensionless variables

$$\sigma = \frac{s}{\ell}, \quad p_1 = \frac{x}{\ell}, \quad p_2 = \frac{y}{\ell}, \quad p_3 = \frac{z}{\ell}, \quad p_4 = \theta, \quad p_5 = \psi, \quad p_6 = \phi, \quad \tau = \omega_0 t, \quad (47)$$

where ω_0 is the reference natural frequency yet to be determined. Moreover, let

$$\begin{cases} \eta_0 = \frac{1}{\rho A \omega_0^2 \ell}, & \xi_{01} = \frac{\kappa G A \eta_0}{\ell}, & \xi_{02} = \frac{E A \eta_0}{\ell}; \\ \eta_1 = \frac{1}{\rho \gamma_{11} \omega_0^2}, & \eta_2 = \frac{1}{\rho \gamma_{22} \omega_0^2}, & \eta_3 = \frac{1}{\rho \gamma_{33} \omega_0^2}, \\ \xi_{11} = \frac{E}{\rho \ell^2 \omega_0^2}, & \xi_{12} = \frac{G \gamma_{33} - E \gamma_{22}}{\rho \ell^2 \gamma_{11} \omega_0^2}, & \xi_{13} = \kappa G A \eta_1, \quad \xi_{14} = E A \eta_1; \\ \xi_{21} = \frac{E}{\rho \ell^2 \omega_0^2}, & \xi_{22} = \frac{G \gamma_{33} - E \gamma_{11}}{\rho \ell^2 \gamma_{22} \omega_0^2}, & \xi_{23} = \kappa G A \eta_2, \quad \xi_{24} = E A \eta_2; \\ \xi_{31} = \frac{G}{\rho \ell^2 \omega_0^2}, & \xi_{32} = \frac{E(\gamma_{22} - \gamma_{11})}{\rho \ell^2 \gamma_{33} \omega_0^2}, & \xi_{33} = \frac{\gamma_{22} - \gamma_{11}}{\gamma_{33}}. \end{cases} \quad (48)$$

Then, substituting the relations (18), (13), (25), (26), (29), (38) and (46) into the nonlinear equation (39), we obtain the governing differential equation of motion in terms of dimensionless displacements $p_i(s, t)$ ($i = 1, 2, \dots, 6$) as follows:

$$\ddot{p}_1 = \{(\xi_{01} - \xi_{02})(-p'_1 \cos p_4 \sin p_5 + p'_2 \sin p_4 - (1 + p'_3) \cos p_4 \cos p_5) \cos p_4 \sin p_5 + \xi_{01} p'_1 - \xi_{02} \cos p_4 \sin p_5\}' + \eta_0 f_x, \quad (49)$$

$$\ddot{p}_2 = \{(\xi_{01} - \xi_{02})(p'_1 \sin p_4 \sin p_5 + p'_2 \cos p_4 + (1 + p'_3) \sin p_4 \cos p_5) \cos p_4 + \xi_{02}(p'_2 + \sin p_4)\}' + \eta_0 f_y, \quad (50)$$

$$\ddot{p}_3 = \{(\xi_{01} - \xi_{02})(-p'_1 \cos p_4 \sin p_5 + p'_2 \sin p_4 - (1 + p'_3) \cos p_4 \cos p_5) \cos p_4 \cos p_5 + \xi_{01}(1 + p'_3) - \xi_{02} \cos p_4 \cos p_5\}' + \eta_0 f_z; \quad (51)$$

$$\begin{aligned} & (\dot{p}_4 \cos p_6 + \dot{p}_5 \sin p_6 \cos p_4)' - (\dot{p}_6 - \dot{p}_5 \sin p_4)(\dot{p}_4 \sin p_6 - \dot{p}_5 \cos p_4 \cos p_6) \\ &= \xi_{11}(p'_4 \cos p_6 + p'_5 \sin p_6 \cos p_4)' - \xi_{12}(p'_6 - p'_5 \sin p_4)(p'_4 \sin p_6 - p'_5 \cos p_6 \cos p_4) \\ & - ((\xi_{13} - \xi_{14})(p'_1 \cos p_4 \sin p_5 - p'_2 \sin p_4 + (1 + p'_3) \cos p_4 \cos p_5) + \xi_{14}) \\ & \cdot \{p'_1(-\sin p_6 \cos p_5 + \cos p_6 \sin p_4 \sin p_5) + p'_2 \cos p_6 \cos p_4 \\ & + (1 + p'_3)(\sin p_6 \sin p_5 + \cos p_6 \sin p_4 \cos p_5)\} \\ & + \eta_1 l_x(\cos p_6 \cos p_5 + \sin p_6 \sin p_4 \sin p_5) + \eta_1 l_y \sin p_6 \cos p_4 \\ & + \eta_1 l_z(-\cos p_6 \sin p_5 + \sin p_6 \sin p_4 \cos p_5), \end{aligned} \quad (52)$$

$$\begin{aligned} & (-\dot{p}_4 \sin p_6 + \dot{p}_5 \cos p_6 \cos p_4)' - (\dot{p}_6 - \dot{p}_5 \sin p_4)(\dot{p}_4 \cos p_6 + \dot{p}_5 \sin p_6 \cos p_4) \\ &= \xi_{21}(p'_5 \cos p_6 \cos p_4 - p'_4 \sin p_6)' - \xi_{22}(p'_6 - p'_5 \sin p_4)(p'_4 \cos p_6 + p'_5 \sin p_6 \cos p_4) \\ & + ((\xi_{23} - \xi_{24})(p'_1 \cos p_4 \sin p_5 - p'_2 \sin p_4 + (1 + p'_3) \cos p_4 \cos p_5) + \xi_{24}) \\ & \cdot \{p'_1(\cos p_6 \cos p_5 + \sin p_6 \sin p_4 \sin p_5) + p'_2 \sin p_6 \cos p_4 \\ & + (1 + p'_3)(-\cos p_6 \sin p_5 + \sin p_6 \sin p_4 \cos p_5)\} \\ & + \eta_2 l_x(-\sin p_6 \cos p_5 + \cos p_6 \sin p_4 \sin p_5) + \eta_2 l_y \cos p_6 \cos p_4 \\ & + \eta_2 l_z(\sin p_6 \sin p_5 + \cos p_6 \sin p_4 \cos p_5), \end{aligned} \quad (53)$$

$$\begin{aligned} & (\dot{p}_6 - \dot{p}_5 \sin p_4)' + \xi_{33}(-\dot{p}_4 \sin p_6 + \dot{p}_5 \cos p_4 \cos p_6)(\dot{p}_4 \cos p_6 + \dot{p}_5 \sin p_6 \cos p_4) \\ &= \xi_{31}(p'_6 - p'_5 \sin p_4)' + \xi_{32}(-p'_4 \sin p_6 + p'_5 \cos p_6 \cos p_4)(p'_4 \cos p_6 + p'_5 \sin p_6 \cos p_4) \\ & + \eta_3(l_x \cos p_4 \sin p_5 - l_y \sin p_4 + l_z \cos p_4 \cos p_5), \end{aligned} \quad (54)$$

where the symbols (\cdot) and $(\cdot)'$ denote differentiation with respect to dimensionless time parameter τ and dimensionless length parameter σ , respectively. The boundary conditions at a fixed end are

$$p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 0. \quad (55)$$

At a free end, the force boundary conditions (44) can be rewritten as

$$\begin{aligned} & \alpha_m \{(\xi_{01} - \xi_{02})(-p'_1 \cos p_4 \sin p_5 + p'_2 \sin p_4 - (1 + p'_3) \cos p_4 \cos p_5) \cos p_4 \sin p_5 \\ & + \xi_{01} p'_1 - \xi_{02} \cos p_4 \sin p_5\} + \frac{\eta_0}{\ell} f_x^b(\tau) = 0, \end{aligned} \quad (56)$$

$$\begin{aligned} & \alpha_m \{(\xi_{01} - \xi_{02})(p'_1 \sin p_4 \sin p_5 + p'_2 \cos p_4 + (1 + p'_3) \sin p_4 \cos p_5) \cos p_4 \\ & + \xi_{02}(p'_2 + \sin p_4)\} + \frac{\eta_0}{\ell} f_y^b(\tau) = 0, \end{aligned} \quad (57)$$

$$\begin{aligned} & \alpha_m \{(\xi_{01} - \xi_{02})(-p'_1 \cos p_4 \sin p_5 + p'_2 \sin p_4 - (1 + p'_3) \cos p_4 \cos p_5) \cos p_4 \cos p_5 \\ & + \xi_{01}(1 + p'_3) - \xi_{02} \cos p_4 \cos p_5\} + \frac{\eta_0}{\ell} f_z^b(\tau) = 0, \end{aligned} \quad (58)$$

while the torque boundary conditions (45) can be rewritten as

$$\begin{aligned} \alpha_m \xi_{11} (p'_4 \cos p_6 + p'_5 \sin p_6 \cos p_4) + \frac{\eta_1}{\ell} l_x^b (\cos p_6 \cos p_5 + \sin p_6 \sin p_4 \sin p_5) \\ + \frac{\eta_1}{\ell} l_y^b \sin p_6 \cos p_4 + \frac{\eta_1}{\ell} l_z^b (-\cos p_6 \sin p_5 + \sin p_6 \sin p_4 \cos p_5) = 0, \end{aligned} \quad (59)$$

$$\begin{aligned} \alpha_m \xi_{21} (p'_5 \cos p_6 \cos p_4 - p'_4 \sin p_6) + \frac{\eta_2}{\ell} l_x^b (-\sin p_6 \cos p_5 + \cos p_6 \sin p_4 \sin p_5) \\ + \frac{\eta_2}{\ell} l_y^b \cos p_6 \cos p_4 + \frac{\eta_2}{\ell} l_z^b (\sin p_6 \sin p_5 + \cos p_6 \sin p_4 \cos p_5) = 0, \end{aligned} \quad (60)$$

$$\alpha_m \xi_{31} (p'_6 - p'_5 \sin p_4) + \frac{\eta_3}{\ell} (l_x^b \cos p_4 \sin p_5 - l_y^b \sin p_4 + l_z^b \cos p_4 \cos p_5) = 0. \quad (61)$$

5. Femlab implementation

Femlab is an open flexible system used to model all types of scientific and engineering problems based on partial differential equations ([Femlab homepage](#)). This gives us freedoms to implement numerical solution procedure of the nonlinear partial differential equations of motion with the corresponding boundary conditions and initial conditions. We introduce the solution procedure of the nonlinear partial differential equations (49)–(54) with the boundary conditions (55) and/or (56)–(61). Let

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{bmatrix}, \quad \mathbf{q} = \dot{\mathbf{p}} = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{p}_4 \\ \dot{p}_5 \\ \dot{p}_6 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \end{bmatrix}, \quad \mathbf{g}^b = \frac{\eta_0}{\ell} \begin{bmatrix} f_x^b(\tau) \\ f_y^b(\tau) \\ f_z^b(\tau) \\ l_1^b(\tau) \\ l_2^b(\tau) \\ l_3^b(\tau) \end{bmatrix}. \quad (62)$$

where

$$\begin{cases} \beta_1 = (\xi_{01} - \xi_{02})(+p'_1 \cos p_4 \sin p_5 - p'_2 \sin p_4 + (1 + p'_3) \cos p_4 \cos p_5) \cos p_4 \sin p_5 \\ \quad - \xi_{01} p'_1 + \xi_{02} \cos p_4 \sin p_5, \\ \beta_2 = (\xi_{02} - \xi_{01})(p'_1 \sin p_4 \sin p_5 + p'_2 \cos p_4 + (1 + p'_3) \sin p_4 \cos p_5) \cos p_4 \\ \quad - \xi_{02} (p'_2 + \sin p_4), \\ \beta_3 = (\xi_{01} - \xi_{02})(p'_1 \cos p_4 \sin p_5 - p'_2 \sin p_4 + (1 + p'_3) \cos p_4 \cos p_5) \cos p_4 \cos p_5 \\ \quad - \xi_{01} (1 + p'_3) + \xi_{02} \cos p_4 \cos p_5, \\ \beta_4 = -\xi_{11} (p'_4 \cos p_6 + p'_5 \sin p_6 \cos p_4), \\ \beta_5 = \xi_{21} (p'_4 \sin p_6 - p'_5 \cos p_6 \cos p_4), \quad \beta_6 = \xi_{31} (-p'_6 + p'_5 \sin p_4). \end{cases} \quad (63)$$

Taking notice of

$$\begin{cases} (\dot{p}_4 \cos p_6 + \dot{p}_5 \sin p_6 \cos p_4)' - (\dot{p}_6 - \dot{p}_5 \sin p_4)(\dot{p}_4 \sin p_6 - \dot{p}_5 \cos p_4 \cos p_6) \\ \quad = \ddot{p}_4 \cos p_6 + \ddot{p}_5 \sin p_6 \cos p_4 - 2\dot{p}_4 \dot{p}_6 \sin p_6 \\ \quad \quad + 2\dot{p}_5 \dot{p}_6 \cos p_4 \cos p_6 - \dot{p}_5^2 \sin p_4 \cos p_4 \cos p_6, \\ (-\dot{p}_4 \sin p_6 + \dot{p}_5 \cos p_6 \cos p_4)' - (\dot{p}_6 - \dot{p}_5 \sin p_4)(\dot{p}_4 \cos p_6 + \dot{p}_5 \sin p_6 \cos p_4) \\ \quad = -\ddot{p}_4 \sin p_6 + \ddot{p}_5 \cos p_6 \cos p_4 - 2\dot{p}_4 \dot{p}_6 \cos p_6 \\ \quad \quad - 2\dot{p}_5 \dot{p}_6 \sin p_6 \cos p_4 + \dot{p}_5^2 \sin p_4 \cos p_4 \sin p_6, \\ (\dot{p}_6 - \dot{p}_5 \sin p_4)' + \xi_{33} (-\dot{p}_4 \sin p_6 + \dot{p}_5 \cos p_4 \cos p_6)(\dot{p}_4 \cos p_6 + \dot{p}_5 \sin p_6 \cos p_4) \\ \quad = (\ddot{p}_6 - \ddot{p}_5 \sin p_4) - \dot{p}_5 \dot{p}_4 \cos p_4 \\ \quad \quad + \xi_{33} (-\dot{p}_4^2 \sin p_6 \cos p_6 + \dot{p}_4 \dot{p}_5 \cos 2p_6 \cos p_4 + \dot{p}_5^2 \sin p_6 \cos p_6 \cos^2 p_4), \end{cases} \quad (64)$$

the nonlinear partial differential equations (49)–(54) can be rewritten as

$$\begin{bmatrix} \dot{\mathbf{p}} \\ \mathbf{M}_a \dot{\mathbf{q}} \end{bmatrix} + \begin{bmatrix} 0 \\ \boldsymbol{\beta}' \end{bmatrix} = \begin{bmatrix} \mathbf{q} \\ \mathbf{g} \end{bmatrix}, \quad (65)$$

where

$$\mathbf{M}_a = \begin{bmatrix} I_{3 \times 3} & \\ & \mathbf{M}_m \end{bmatrix}, \quad \mathbf{M}_m = \begin{bmatrix} \cos p_6 & \sin p_6 \cos p_4 & 0 \\ -\sin p_6 & \cos p_6 \cos p_4 & 0 \\ 0 & -\sin p_4 & 1 \end{bmatrix}, \quad (66)$$

and

$$\begin{cases} g_1 = \eta_0 f_x, & g_2 = \eta_0 f_y, & g_3 = \eta_0 f_z; \\ g_4 = -\xi_{12}(p'_6 - p'_5 \sin p_4)(p'_4 \sin p_6 - p'_5 \cos p_6 \cos p_4) \\ \quad - ((\xi_{13} - \xi_{14})(p'_1 \cos p_4 \sin p_5 - p'_2 \sin p_4 + (1 + p'_3) \cos p_4 \cos p_5) + \xi_{14}) \\ \quad \cdot \{p'_1(-\sin p_6 \cos p_5 + \cos p_6 \sin p_4 \sin p_5) + p'_2 \cos p_6 \cos p_4 \\ \quad + (1 + p'_3)(\sin p_6 \sin p_5 + \cos p_6 \sin p_4 \cos p_5)\} + \eta_1 l_1 \\ \quad + 2\dot{p}_4 \dot{p}_6 \sin p_6 - 2\dot{p}_5 \dot{p}_6 \cos p_4 \cos p_6 + \dot{p}_5^2 \sin p_4 \cos p_4 \cos p_6, \\ g_5 = -\xi_{22}(p'_6 - p'_5 \sin p_4)(p'_4 \cos p_6 + p'_5 \sin p_6 \cos p_4) \\ \quad + ((\xi_{23} - \xi_{24})(p'_1 \cos p_4 \sin p_5 - p'_2 \sin p_4 + (1 + p'_3) \cos p_4 \cos p_5) + \xi_{24}) \\ \quad \cdot \{p'_1(\cos p_6 \cos p_5 + \sin p_6 \sin p_4 \sin p_5) + p'_2 \sin p_6 \cos p_4 \\ \quad + (1 + p'_3)(-\cos p_6 \sin p_5 + \sin p_6 \sin p_4 \cos p_5)\} + \eta_2 l_2 \\ \quad + 2\dot{p}_4 \dot{p}_6 \cos p_6 + 2\dot{p}_5 \dot{p}_6 \sin p_6 \cos p_4 - \dot{p}_5^2 \sin p_4 \cos p_4 \sin p_6, \\ g_6 = \xi_{32}(-p'_4 \sin p_6 + p'_5 \cos p_6 \cos p_4)(p'_4 \cos p_6 + p'_5 \sin p_6 \cos p_4) + \eta_3 l_3 + \dot{p}_5 \dot{p}_4 \cos p_4 \\ \quad + \xi_{33}(\dot{p}_4^2 \sin p_6 \cos p_6 - \dot{p}_4 \dot{p}_5 \cos 2p_6 \cos p_4 - \dot{p}_5^2 \sin p_6 \cos p_6 \cos^2 p_4). \end{cases} \quad (67)$$

Using the general form for nonlinear PDEs with 12 dependent variables, Eq. (65) can be implemented in Femlab in terms of the Matlab code (for details, see Comsol, 2004). The Dirichlet boundary conditions should be adopted at a fixed end, i.e., $\mathbf{p} = 0$. At a free end, the Neumann boundary conditions should be adopted, i.e., $\alpha_m \boldsymbol{\beta} = -\mathbf{g}^b$, where $\boldsymbol{\beta}$ and \mathbf{g}^b are defined by (62).

For the formulated nonlinear model of a Cosserat rod, the numerical analysis using the Femlab programming language at the Matlab command line consists of the following steps:

1. Formulation of the governing equations;
2. Discretization of the equations;
3. Solution of the equations;
4. Interpretation of the results.

Given the one-dimensional geometry data, an initial finite element mesh is easily generated by *meshinit*. The *meshextend* command is used for discretization of the PDE problem and can be modified to improve accuracy. The geometry, nonlinear PDEs and boundary conditions are defined by a set of fields in the Matlab codes. For solving purposes Femlab contains specific solvers (like static, dynamic, linear, nonlinear solvers) for specific PDE problems. Here, we use *femtime* to solve the dynamic problem of the system.

6. Simulation results and discussion for a simple cantilever

A cantilever (fixed at the left hand and free at the right hand), as shown in Fig. 2, is now presented as a simple example to demonstrate the numerical solution procedure of the proposed Cosserat rod model. Numerical calculations based on (65) are carried out for the uniform horizontal cantilever of length $\ell = 1.0$ m, of constant cross-section with width $b = 0.06$ m and thickness $a = 0.04$ m. The mass density, the Young's modulus and the Poisson's ratio are assumed to be $\rho = 2.730 \times 10^3$ kg/m³, $E = 7.1 \times 10^{10}$ Pa and $\nu = 0.32$, respectively. The shear correction factor for rectangular cross-section is taken to be $\kappa = 0.833$. The reference frequency is

chosen to be the lowest natural frequency of the linearized system of a cantilever without considering the effect of shearing deformations, i.e.,

$$\omega_0 = \frac{1.875^2}{\ell^2} \sqrt{\frac{E\gamma_{22}}{\rho A}} = \frac{1.875^2}{\ell^2} \sqrt{\frac{Ea^2}{12\rho}} = 207.0236 \text{ rad s}^{-1}.$$

Based on the derived nonlinear system (65), numerical simulations are performed to investigate the dynamic responses of the cantilever under harmonic excitations. The differential equations of motion are full coupled by the nonlinear terms and could exhibit internal resonance introduced by the nonlinearities. They also exhibit external resonances when the external excitation is periodic and the frequency of a component of its Fourier series is near one of the natural frequencies of the system, or near a multiple of a natural frequencies. The detailed analysis of complex dynamic behavior, such as bifurcation (Vu-Quoc and Li, 1995) and chaos, of the system is not the main focus of this paper. We only present here the simulation results of the dynamic responses of the system under external harmonic excitations.

The dynamical responses of the free end of the cantilever (with zero initial displacements and velocities) under external distributing loads

$$f_x(t) = 2 \sin(\pi s) \sin(8\omega_0 t) \text{ kN m}^{-1}, \quad f_y(t) = 2 \sin(\pi s) \sin(8\omega_0 t) \text{ kN m}^{-1},$$

are obtained using the Femlab/Matlab, and shown in Fig. 4. It is observed that the transverse displacements $x(1, t)$ and $y(1, t)$ consist of more than one harmonic motions with different frequencies. The longitudinal displacement $z(1, t)$ and the torsional oscillation $\phi(1, t)$ are obviously not periodic motions, which is not surprising, since various dynamic phenomena are expected for such a strongly nonlinear and fully coupling system. Moreover, it can be observed that the longitudinal displacement $z(1, t)$ is a higher order small term comparing with other variables like $x(1, t)$ and $y(1, t)$. The transverse displacement time histories $x(s, t)$ and $y(s, t)$ of the nonlinear coupling model under external harmonic excitations are shown in Figs. 5 and 6, respectively.

Dividing the cantilever into 10 Cosserat Rod Elements (CRE) of equal length, see Cao et al. (2006) for the formulation of CREs, one can establish the nonlinear ordinary differential equations of motion for solving the free displacements. Numerical simulations for the responses of the model under the same external harmonic

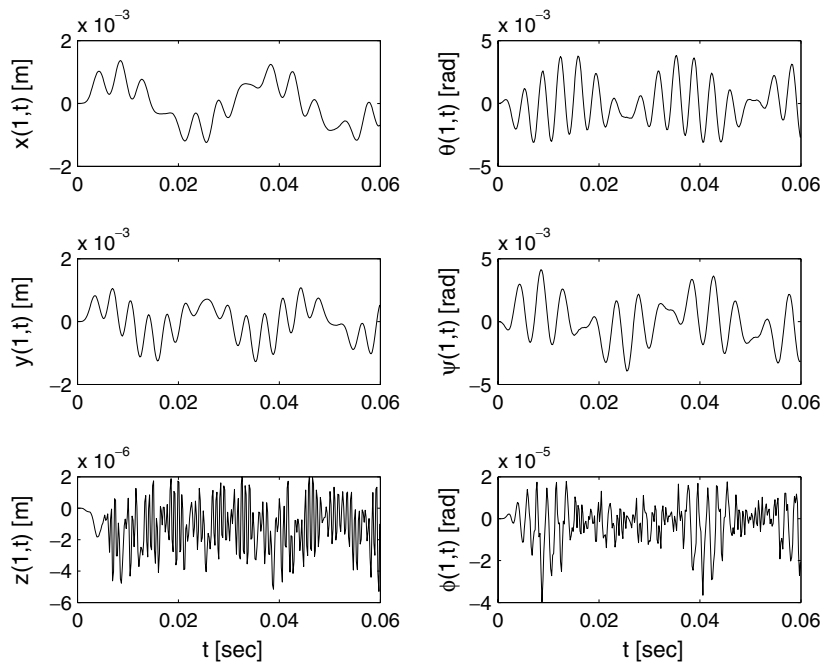


Fig. 4. Dynamical responses of the cantilever at the free end under harmonic excitations.

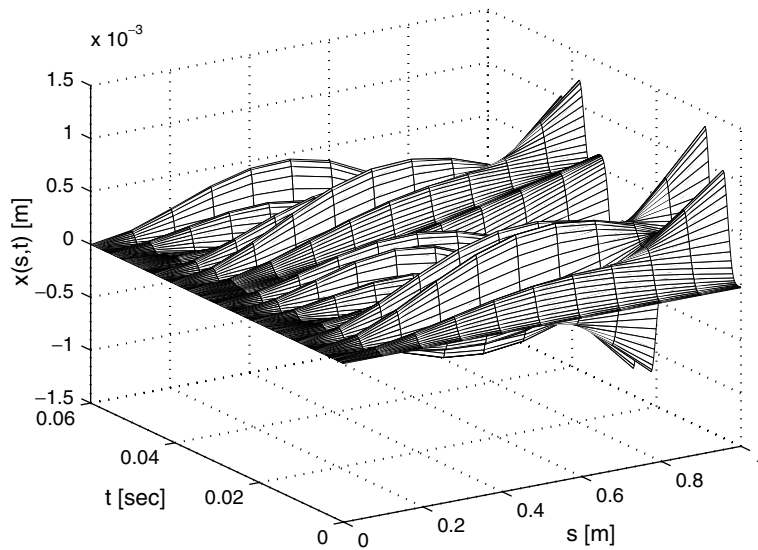


Fig. 5. Displacement time histories of the cantilever $x(s,t)$ [m] under harmonic excitations.

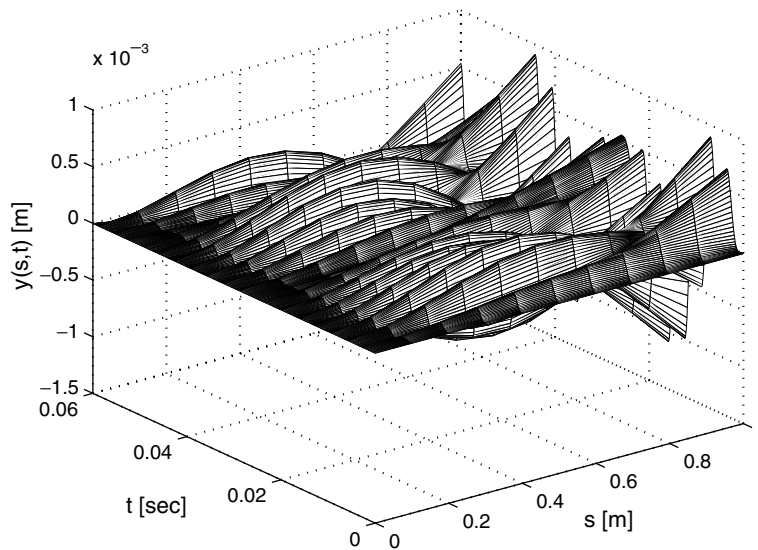


Fig. 6. Displacement time histories of the cantilever $y(s,t)$ [m] under harmonic excitations.

excitations are performed with Matlab. The corresponding dynamical responses of the CRE model are depicted in Figs. 7–9 for a comparison of the results in Figs. 4–6 obtained in this paper.

7. Conclusions

The nonlinear partial differential equations of motion for slender elastic rods have been formulated using the Cosserat theory. Unlike other formulations presented in the literature, the nonlinear dynamic model formulated here are explicitly in terms of the displacements and angular variables. Flexure along two principal axes, extension, shear and torsion are considered, and care is taken to account for all the nonlinearities include contributions from both the curvature expression and inertia terms.

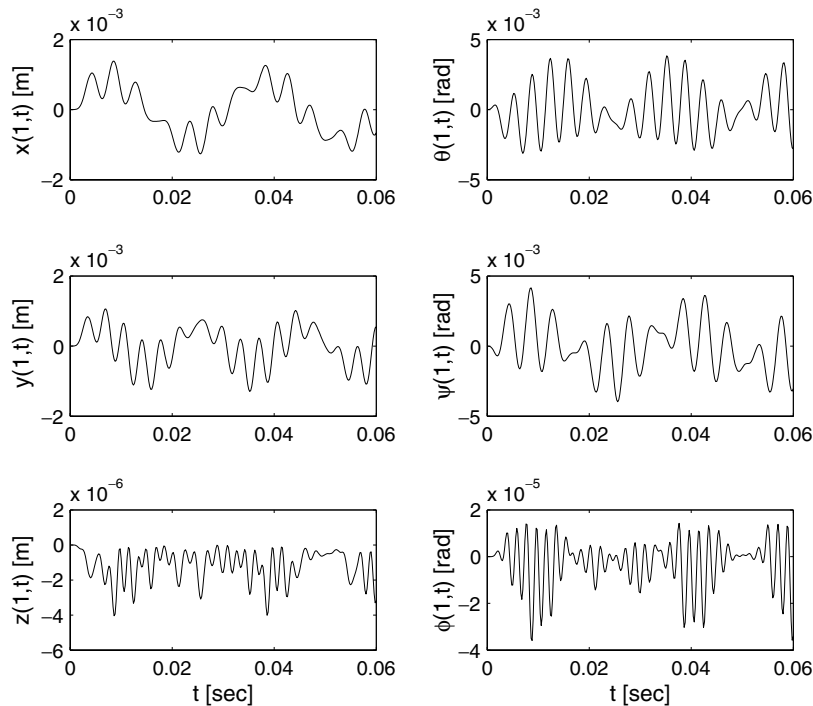


Fig. 7. Dynamical responses of the cantilever at the free end under harmonic excitations: Cosserat Rod Element Approach.

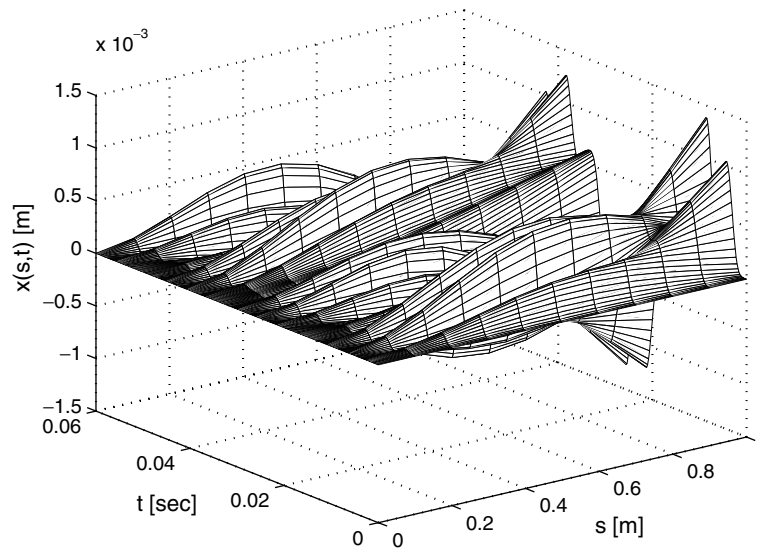


Fig. 8. Displacement time histories of the cantilever $x(s,t)$ [m] under harmonic excitations: Cosserat Rod Element Approach.

A numerical solution procedure in terms of the Femlab/Matlab interfaces has been presented for numerically solving the formulated nonlinear dynamical model with the corresponding boundary and initial conditions. A simple cantilever under external harmonic excitations has been presented to demonstrate the use of the formulation developed, and to illustrate the numerical solution procedure.

Based on the case study of applying the proposed modelling strategy for nonlinear dynamics of elastic rods, the following conclusions have been drawn.

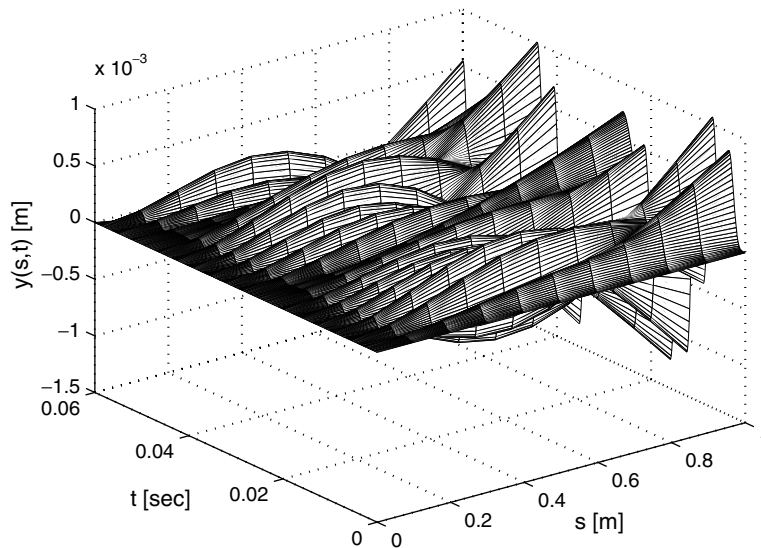


Fig. 9. Displacement time histories of the cantilever $y(s,t)$ [m] under harmonic excitations: Cosserat Rod Element Approach.

1. The explicit nature of the construction permits us to directly express the nonlinear dynamic model of the elastic rod and the corresponding boundary conditions in terms of the displacements and angular variables, thus the dynamical analysis of elastic rods can be carried out rather simply.
2. The governing partial differential equations of motion can be easily expanded to contain nonlinearities up to order three to render them amenable to the study of moderately large amplitude flexural-torsional-shearable oscillations by perturbation techniques. Moreover, the approach on dimension reduction of dynamical systems, such as the nonlinear Galerkin method (Rega and Troger, 2005) and the Karhunen–Loeve method (which in the mathematical literature is also called Proper Orthogonal Decomposition (POD) method (Georgiou, 2005)), could be used to develop reduced-order dynamic models of the coupled nonlinear partial differential systems derived here. Therefore, the full coupled nonlinear model formulated here could be very useful for the detailed analysis of complex dynamic behavior, such as bifurcation and chaos, of the slender rods.
3. The nonlinear dynamic model could be numerically solved in terms of the Femlab/Matlab interfaces. The Femlab environment offers a powerful simulation tool for analyzing the responses of nonlinear dynamic models (of slender material elements such as those found in an MEMS device) presented here.

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